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**LETTER TO THE EDITOR**

# Localization properties of the periodic random Anderson model

M Hilke†

Université de Genève, Département de Physique Théorique, 24 Quai Ernest-Ansermet CH-1211 Genève 4, Switzerland

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**Abstract.** We consider diagonal disordered one-dimensional Anderson models with an underlying periodicity. We assume the simplest periodicity, i.e. we have essentially two lattices, one that is composed of the random potentials and the other of non-random potentials. Due to the periodicity, special resonance energies appear, which are related to the lattice constant of the non-random lattice. Further on two different types of behaviour are observed at the resonance energies. When a random site is surrounded by non-random sites, this model exhibits *extended* states at the resonance energies, whereas otherwise all states are localized with, however, an increase of the localization length at these resonance energies. We study these resonance energies and evaluate the localization length and the density of states around these energies.

Localization properties of disordered systems were first examined in tight-binding models by Anderson [1], who showed that certain states are localized due to disorder. His result was generalized by Mott and Twose [2] and Landauer [3] who conjectured and later several authors [4, 5] proved that, in one dimension, all states are localized for any amount of disorder.

Recently, however, there have been a number of experiments in quasi-one-dimensional systems which exhibit unusual high conductivities. These systems are polymers as well as mesoscopic rings [6–8]. It seems, therefore, of great importance to study delocalization mechanism in disordered systems. As has already been pointed out in some recent works on disordered systems, correlations in disorder can be a driving force for delocalization in one dimension [6, 9–14] and in two dimensions [15]. The new approach in this paper is to consider systematically the effect of a deterministic periodic potential, as a source of correlations in the disorder.

In this study we consider tight-binding models related to the original Anderson model. The periodicity is introduced by considering two underlying lattices, of which one is composed by the random sites and the other by the deterministic sites. In addition, we suppose that all deterministic sites are constant. This letter is divided in two parts. In the first part we consider the special case where each random site is surrounded by at least one constant neighbour site. In this case, as discussed later, there exist discrete resonance energies for which the states are *overall extended*, i.e. with an infinite localization length. In the second part, where the restriction above does not apply, we find the same resonance

† Current address: Department of Electrical Engineering, Princeton University, NJ-08544, USA.

energies. The only *essential* difference is that instead of having an infinite localization length these states present only an enhancement of this length at these critical energies.

In the usual diagonal disordered Anderson model with uncorrelated disorder, on each site the localization length can be evaluated and yields $L_c(E) = 24(4-E^2)/W^2$ [16], where $W$ is the width of the disorder potential distribution, for small disorder. The consequence is that all states are localized for this model. This is, however, only true if we take the average over different configurations of impurities, as otherwise it is well known that so-called Azbel [17] resonances can appear for a given configuration. These resonances of extended states, however, disappear when we average over different configurations.

For the first part we start with the following Anderson model

$$(V_l - \epsilon)\Psi_l + \Psi_{l+1} + \Psi_{l-1} = 0 \tag{1}$$

where $V_l$ is non-zero only if $l$ is a multiple of $d$, where $d$ is an integer, i.e. $V_{dl}$ are random and $V_{dl+1} = \cdots = V_{d(l+1)-1} = 0$. The case where the deterministic sites are non-zero but constant, is trivially obtained by shifting the energy.

The method used to solve this model was developed by Erdös and Herndon [18] and later simplified by Felderhof [19]. The idea is as follows. We suppose that between impurities the solution can be written as the sum of an incident plane wave and a reflected plane wave, i.e.

$$\Psi_l = A_n\,\mathrm{e}^{\mathrm{i}kl} + B_n\,\mathrm{e}^{-\mathrm{i}kl} \qquad X_{n-1} < l < X_n \tag{2}$$

where $X_n$ are the positions of the impurities with value $V_{dn}$. Inserting this in (1) yields the following transfer matrix relation for $d > 1$,

$$\begin{pmatrix} A_{n+1} \\ B_{n+1} \end{pmatrix} = \begin{pmatrix} \alpha_n & \mathrm{e}^{-2\mathrm{i}kX_n}(-\mathrm{i}W_n) \\ \mathrm{e}^{2\mathrm{i}kX_n}(\mathrm{i}W_n) & \alpha_n^* \end{pmatrix} \begin{pmatrix} A_n \\ B_n \end{pmatrix} \tag{3}$$

where $W_n = V_{dn}/2\sin k$, $\alpha_n = 1 + \mathrm{i}W_n$ and $2\cos k = \epsilon$. Instead of considering this transfer matrix, Felderhof uses the 3-vector transfer matrix, namely

$$\Gamma_n = \begin{pmatrix} 1 - W_n^2 - 2\mathrm{i}W_n & -W_n^2 - \mathrm{i}W_n & -W_n^2 \\ 2W_n^2 + 2\mathrm{i}W_n & 1 + 2W_n^2 & 2W_n^2 - 2\mathrm{i}W_n \\ -W_n^2 & -W_n^2 + \mathrm{i}W_n & 1 - W_n^2 + 2\mathrm{i}W_n \end{pmatrix} \tag{4}$$

and

$$G_n = \begin{pmatrix} \mathrm{e}^{2\mathrm{i}k(X_n - X_{n-1})} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \mathrm{e}^{-2\mathrm{i}k(X_n - X_{n-1})} \end{pmatrix}. \tag{5}$$

The main result of Felderhof is to obtain $R/T = (P(2,2) - 1)/2$, where $R$ is the reflection coefficient and $T$ the transmission coefficient and

$$P = \Gamma_n \cdot G_n \cdot \Gamma_{n-1} \cdot G_{n-1} \ldots . \tag{6}$$

There are essentially two cases which can be solved analytically in this approach. The first one was studied by Felderhof, who considered the limit $k \to \infty$ and found that the average resistance grows exponentially with the number of scatterers or Anderson localization. In our case we only consider the band centre, i.e. where $E = 0$ and when $d$ is even we have, as $E = 2\cos k$ and as $X_n - X_{n-1} = d$, that $G_n = I$, where $I$ is the identity matrix. Calculating (6) yields the surprisingly simple expression

$$P(2,2) = 2\left(\sum_{n=1}^{N} W_n\right)^2 + 1. \tag{7}$$

This shows that when the sum of the impurities is zero the reflection coefficient vanishes. This result can be extended to the case where $w = \sum_{n=1}^{N} W_n$ is non-zero by redefining $E = \epsilon - w/N = 2\cos k$, which ensures that $\sum_n (W_n - w/N) = 0$ and implies that the reflection coefficient vanishes when $\epsilon = w/N$. This result states that we have *total transmission* for this model. In fact if we suppose that the average of $V_n = 0$ then $w/N \to 0$ for $N \to \infty$, due to the central limit theorem. This, therefore, implies that we have total transmission at the band centre in the thermodynamic limit.

Above we showed that we get total transmission at the critical energy. It is straightforward to see that the state at $\epsilon = 0$ is *overall extended*, as one only needs to suppose that $\Psi_{dl} = 0$ and one is left with an ordered Anderson model. In the following we study the dependence on energy of the localization length around the critical energy. Starting again from equation (1) and for $d$ even, we renormalize this equation as follows,

$$(W_{2l+1} - \epsilon)\Psi_{2l-2} + \Omega_{2l}(\epsilon)\Psi_{2l} + (W_{2l-1} - \epsilon)\Psi_{2l+2} = 0 \tag{8}$$

where

$$\Omega_{2l}(\epsilon) = W_{2l+1} + W_{2l-1} - 2\epsilon - (W_{2l+1} - \epsilon)(W_{2l} - \epsilon)(W_{2l-1} - \epsilon). \tag{9}$$

Furthermore, in our diluted model $W_{2l+1} = 0$, which when inserted in (9), yields

$$\Psi_{2l-2} + (2 + \epsilon(W_{2l} - \epsilon))\,\Psi_{2l} + \Psi_{2l+2} = 0 \tag{10}$$

for $\epsilon \neq 0$. This last model was extensively studied in the limit $\epsilon \ll 1$ by Derrida and Gardner [20]. They calculated the complex Lyapounov exponent $\gamma$, where the real part corresponds to the inverse localization length and the imaginary part to $\pi$ times the integrated density of states. Their results can be expressed as follows,

$$\mathrm{Re}(\gamma) \simeq K_1 \epsilon^{2/3} \langle W^2 \rangle^{1/3} \qquad \mathrm{Im}(\gamma) \simeq K_2 \epsilon^{2/3} \langle W^2 \rangle^{1/3} \tag{11}$$

where $K_1 = 0.29\ldots$ and $K_2 = 0.16\ldots$ and $\langle\cdot\rangle$ is the average over all impurities. From (11) it is straightforward that the inverse localization length $L_\mathrm{c}^{-1}$ scales as

$$L_\mathrm{c}^{-1} \sim \epsilon^{2/3} \langle W^2 \rangle^{1/3} \tag{12}$$

and the density of states is

$$\rho(\epsilon) = \partial_\epsilon \,\mathrm{Im}\,\gamma(\epsilon) \sim \epsilon^{-1/3}. \tag{13}$$

Above we only considered the extended states at $\epsilon = 0$ for $d$ even, but there exist, in general, $d - 1$ energies at which the states are extended. For $d = 3$, for example, we have delocalized states for $\epsilon = -1$ and $\epsilon = 1$. For any $d$ they can be easily evaluated as they are the roots of $(1, 0) \cdot T_d \cdot \binom{1}{0} = 0$, where

$$T_d = \prod_{n=1}^{d-1} \begin{pmatrix} \epsilon & -1 \\ 1 & 0 \end{pmatrix}. \tag{14}$$

The solutions can be written as $\epsilon = 2\cos n\pi/d$, where $n$ is an integer with $|n| < d$.

These critical energies correspond to the resonance energies discussed by Derrida and Gardner [20], for which their expansion in the low disorder limit is non-trivial. The two uppermost curves in figure 1 show the localization length as a function of energy. One clearly sees the infinite localization length at the critical energies.

In the following we study numerically the case where we can have two neighbouring random sites. In general, one obtains a similar result as that for the completely random model, where in a first-order perturbation $L_\mathrm{c} \sim (4 - E^2)$ [16]. The changes occur at the resonance energies discussed above. In fact at these energies we have an enhancement of the localization length, as shown in figure 1 for the two lowest curves.

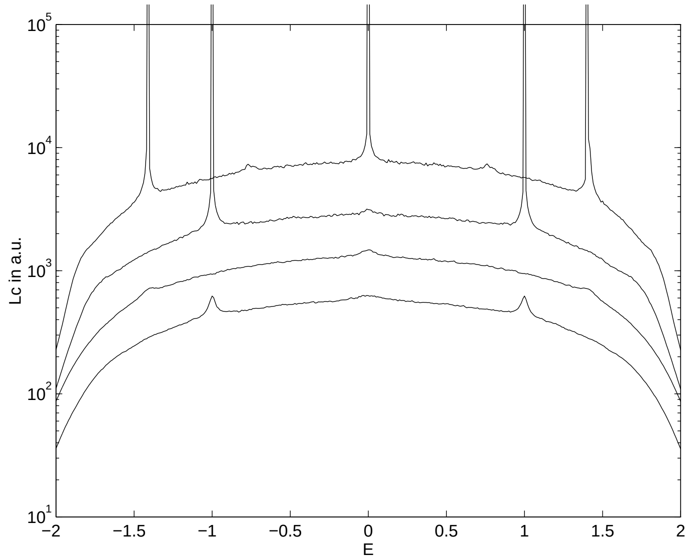


**Figure 1.** Localization length in arbitrary units as a function of energy. The two uppermost curves are from the case discussed in the first part of this letter with extended states and $d = 4$ and $d = 3$, respectively. The two lowest curves represent the case where the localization length is not infinite but enhanced at the resonance energies with $d = 4$ and $d = 3$, respectively.

It is interesting to note that the peaks of the localization length correspond exactly to the resonance energies discussed above. The relative enhancement, however, decreases with increasing $d$ and in the limit $d \longrightarrow \infty$ we recover the usual uncorrelated result. The plot is shown for a system size of 1000 and averaged over a thousand configurations. $d = 3$ corresponds to the case where every third site is non-random and the sites in between are random. This is opposite to the case discussed in the first part of this letter, and shown in the second uppermost curve of figure 1, where every third site is random and the sites in between are non-random.

This last study demonstrates that a periodic correlation in the disorder is not enough in order to completely delocalize some states. This correlation enhances the localization length at some energies related to the periodicity. It appears that an important factor is the isolation of the random sites. For different models, however, like the dimer or multi-mer case [9–15], this is not an essential condition.

The main conclusion we can derive from this study is that if we consider an Anderson model with every $d$'s site disordered instead of each site, where $d$ is an integer and $d \geqslant 2$, the model exhibits extended states at some critical energies. The exponents describing the strength of the divergence remain the same for the different energies, i.e. $\nu = 2/3$. The delocalization properties of these diluted random systems can be understood in terms of correlations, as diluting the system is equivalent to introducing a long-range periodic correlation in the disorder. Outside of the critical energies this dilute Anderson model has the same localization properties as the usual one. When we equate the localization length with the size of the system, in order to estimate the number of states whose localization length exceeds the system size, we observe using (13) that this number is independent of the size; therefore, in the infinite size limit, these states should not have any influence on the transport properties. However, for small quasi-one-dimensional systems like, for example, disordered superlattices of heterostructures or systems with very few impurities these effects do influence the transport properties. Finite temperatures can also reduce the effective system size and lead to changes in the transport properties. The results presented above have important consequences on discretization procedures of disordered systems. Indeed, for a given number of disordered sites, the choice of the elementary lattice constant drastically affects the localization properties of the system.

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